

A simple construction of indecomposable higher Chow cycles in elliptic surfaces

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Dedicated to the 60th birthday of Professor James D. Lewis

1 Introduction

Let $\mathrm{CH}^j(X, i)$ be Bloch's higher Chow groups of a projective smooth variety X over \mathbb{C} . A higher Chow cycle $z \in \mathrm{CH}^j(X, i)$ is called *indecomposable* if it does not belong to the image of the map of the product

$$\mathrm{CH}^1(X, 1) \otimes \mathrm{CH}^{j-1}(X, i-1) \longrightarrow \mathrm{CH}^j(X, i).$$

Of particular interest to us is $\mathrm{CH}^2(X, 1)$. For $A = \mathbb{Q}$ or \mathbb{R} , we say $z \in \mathrm{CH}^2(X, 1)$ *A-regulator indecomposable* if the regulator class $\mathrm{reg}(z) \in H_{\mathcal{D}}^3(X, A(2))$ in the Deligne-Beilinson cohomology group with coefficients in A does not belong to the image of $H_{\mathcal{D}}^1(X, \mathbb{Z}(1)) \otimes H_{\mathcal{D}}^2(X, A(1)) \cong \mathbb{C}^\times \otimes \mathrm{CH}^1(X) \otimes A$. In other words, z is *A-regulator indecomposable* if and only if

$$\mathrm{reg}(z) \neq 0 \in \mathrm{Ext}_{\mathrm{MHS}}^1(A, H_{\mathrm{ind}}^2(X, A(2))), \quad H_{\mathrm{ind}}^2(X, A) := H^2(X, A)/\mathrm{NS}(X) \otimes A.$$

Obviously \mathbb{R} -reg. indecomp. \implies \mathbb{Q} -reg. indecomp. \implies indecomposable.

Quite a lot of examples of \mathbb{Q} or \mathbb{R} -regulator indecomposable cycles are obtained by many people ([1], [3], [4], [5], [6] and more).

In this note we construct \mathbb{R} -regulator indecomposable cycles for X an elliptic surface which satisfies certain conditions. The main theorem is the following.

Theorem 1.1 *Let S be a smooth irreducible curve over \mathbb{C} . Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightleftharpoons[s]{f} & \mathcal{C} \\ g \searrow & & \swarrow h \\ & S & \end{array}$$

be an elliptic fibration over S with a section s . This means that g and h are projective smooth morphisms of relative dimension 2 and 1 respectively, and the general fiber of f is an elliptic curve. For a point $t \in S$ we denote $X_t = g^{-1}(t)$ or $C_t = h^{-1}(t)$ the fibers over t . Assume that the following conditions hold.

- (1) Let η be the generic point of S . Then there is a split multiplicative fiber $D_\eta = f^{-1}(P) \subset X_\eta$ of Kodaira type I_n , $n \geq 1$.
- (2) Let $\mathcal{D} \subset \mathcal{X}$ be the closure of D_η . Then there is a closed point $0 \in S(\mathbb{C})$ such that the specialization $D_0 := \mathcal{D} \times_{\mathcal{X}} X_0$ is multiplicative of type I_m with $m > n$.

Then the composition

$$\mathrm{CH}^1(D_t, 1) \longrightarrow \mathrm{CH}^2(X_t, 1) \xrightarrow{\mathrm{reg}} \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{R}, H^2(X_t, \mathbb{R}(2)) / \mathrm{NF}(X_t) \otimes \mathbb{R})$$

is non-zero for a general $t \in S(\mathbb{C})$. Here $\mathrm{NF}(X_t) \subset \mathrm{NS}(X_t)$ denotes the subgroup generated by components of singular fibers and the section $s(C_t)$. In particular, if $\mathrm{NF}(X_t) \otimes \mathbb{Q} = \mathrm{NS}(X_t) \otimes \mathbb{Q}$, then there is a \mathbb{R} -regulator indecomposable higher Chow cycle.

The key assumption is a “degeneration of D_t ”, which often appears in a family of elliptic surfaces.

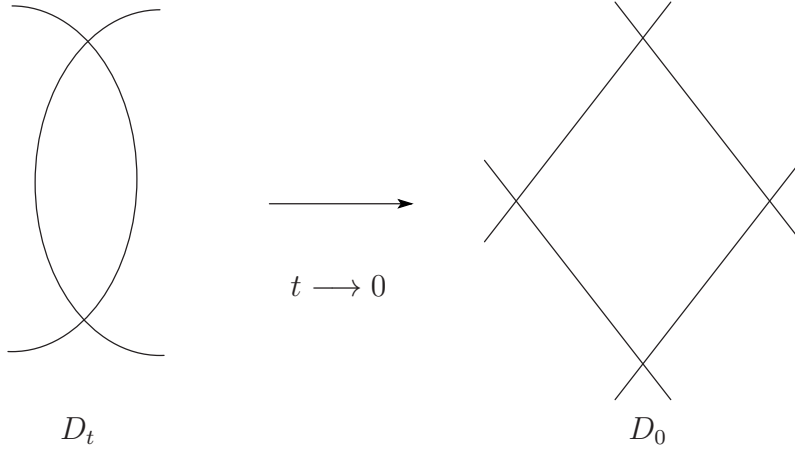


Figure : Degeneration of I_2 to I_4

In §3, we will apply Theorem 1.1 to construct a \mathbb{R} -reg. indecomp. cycle in a self-product of elliptic curves. However, to do it in more general situation, the computation of the Picard number might be an obstacle. Indeed it is easy to compute the rank of $\mathrm{NF}(X_t)$, whereas there is no general method to do it for $\mathrm{NS}(X_t)$, and usually it is done by case-by-case analysis.

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2 Proof of Theorem 1.1

We keep the notation and assumption in Theorem 1.1.

2.1 Step 1 : Construction of a higher Chow cycle

Lemma 2.1 *Let $f : X \rightarrow C$ be an elliptic fibration over a field K of characteristic $\neq 2, 3$, with a (fixed) section s . Let $D = f^{-1}(P)$ be a split multiplicative fiber, and let $i : D \hookrightarrow X$. Then there is an exact sequence*

$$\mathrm{CH}^1(\tilde{D}, 1)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^1(D, 1)_{\mathbb{Q}} \xrightarrow{v} \mathbb{Q} \longrightarrow 0 \quad (2.1)$$

where $\tilde{D} \rightarrow D$ is the normalization. There exists a higher Chow cycle $Z \in \mathrm{CH}^2(D, 1)$ such that $v(Z) \neq 0$ and $i_*(Z) \in \mathrm{CH}^2(X, 1)$ is vertical to generators of $\mathrm{NF}(X)$. Here “vertical to E ” means that it lies in the kernel of the composition $\mathrm{CH}^2(X, 1) \rightarrow \mathrm{CH}^2(\tilde{E} \times_K \overline{K}, 1) \rightarrow \overline{K}^{\times} \otimes \mathbb{Q}$.

Proof. We omit to show the exact sequence (2.1) (easy exercise). We show the existence of Z . In this proof, we use K -groups rather than higher Chow groups. Let $D = \sum_i D_i$ be the irreducible decomposition. It is enough to construct $Z \in K'_1(D)^{(1)}$ such that $v(Z) \neq 0$ and $i_*(Z)$ is vertical to each D_i and $s(C)$ because it is obviously vertical to the other fibral divisors. Since D is split multiplicative, each D_i is geometrically irreducible and the singularities of D are K -rational. Therefore we may assume $K = \overline{K}$ by the standard norm argument.

It is enough to show that the image of the composition

$$K'_1(D)^{(1)} \xrightarrow{i_*} K_1(X)^{(2)} \xrightarrow{i^*} K_1(\tilde{D})^{(2)} \cong (K^{\times} \otimes \mathbb{Q})^{\oplus n}$$

coincides with that of

$$K'_1(\tilde{D})^{(1)} \xrightarrow{i_*} K_1(X)^{(2)} \xrightarrow{i^*} K_1(\tilde{D})^{(2)} \cong (K^{\times} \otimes \mathbb{Q})^{\oplus n}.$$

Indeed, the above implies that there is a cycle $Z_0 \in K'_1(D)^{(1)}$ such that $v(Z_0) \neq 0$ and $i_*(Z_0)$ is vertical to each D_i . Let $f^* : K_1(K) \rightarrow K'_1(D)^{(1)}$. Then $Z := Z_0 + f^*(\lambda)$ for suitable $\lambda \in K^{\times}$ can be vertical to the section $s(C)$ and D_i . Moreover $v(Z) = v(Z_0) \neq 0$.

To do the above we may replace X with $\hat{X} = f^{-1}(\mathrm{Spec} K[[s]])$ the formal neighborhood around D . Then it is enough to show that there is $Z_0 \in K'_1(D)^{(1)}$ such that $v(Z_0) \neq 0$ and $i_*(Z_0) = 0$ in $K_1(\hat{X})^{(2)}$. To do this, we may further replace \hat{X} with \hat{X}_n the minimal desingularization of $\hat{X} \times \mathrm{Spec} K[[s^{1/n}]]$ for some $n \geq 1$ due to a commutative diagram

$$\begin{array}{ccccc} \mathbb{Q} & \xleftarrow{v} & K'_1(D_n)^{(1)} & \xrightarrow{i_*} & K_1(\hat{X}_n)^{(2)} \\ \downarrow n & & \downarrow \phi_* & & \downarrow \phi_* \\ \mathbb{Q} & \xleftarrow{v} & K'_1(D)^{(1)} & \xrightarrow{i_*} & K_1(\hat{X})^{(2)} \end{array}$$

where $\phi : \hat{X}_n \rightarrow \hat{X}$. Thus we can assume \hat{X} is defined by a Weierstrass equation

$$y^2 = x^3 + x^2 + c(s), \quad c(s) \in sK[[s]].$$

Then letting $\partial : K_2(\hat{X} \setminus D)^{(2)} \rightarrow K'_1(D)^{(1)}$ be the boundary map, we put

$$Z_0 := \partial \left\{ \frac{y-x}{y+x}, \frac{-c}{x^3} \right\}.$$

This satisfies $v(Z_0) \neq 0$ and $i_*(Z_0) = 0$ in $K_1(\hat{X})^{(2)}$. □

By Lemma 2.1, there is a higher Chow cycle $Z_\eta \in \text{CH}^1(D_\eta, 1)$ such that $v(Z_\eta) \neq 0$ and it is vertical to $\text{NF}(X_\eta)$. We use the same symbol “ Z_η ” for $i_*(Z_\eta) \in \text{CH}^2(X_\eta, 1)$ with $i : D_\eta \hookrightarrow X_\eta$ since it will be clear from the context which is meant.

Let $\mathcal{Z}^* \subset \mathcal{X}^*$ be the closure of Z_η in $\mathcal{X}^* := g^{-1}(S^*)$ for some nonempty Zariski open $S^* \subset S$. Then the goal is to show nonvanishing

$$\text{reg}(Z_t) \neq 0 \in \text{Ext}_{\text{MHS}}^1(\mathbb{R}, H^2(X_t, \mathbb{R}(2))), \quad Z_t := \mathcal{Z}^*|_{X_t} \quad (2.2)$$

for a general $t \in S^*(\mathbb{C})$. Indeed, since Z_t is vertical to $\text{NF}(X_t)$, the above implies the desired nonvanishing

$$\text{reg}(Z_t) \neq 0 \in \text{Ext}_{\text{MHS}}^1(\mathbb{R}, H^2(X_t, \mathbb{R}(2))/\text{NF}(X_t) \otimes \mathbb{R}).$$

2.2 Step 2 : Boundary of Z_η

Lemma 2.2 *Let $\partial : \text{CH}^2(X_\eta, 1) \rightarrow \text{CH}^1(X_0)$ be the boundary map arising from the localization exact sequence. Then $\partial(Z_\eta)$ is non-torsion.*

Proof. We may assume $S = \text{Spec} R$ where R is a DVR, with a closed point 0 and generic point η . Let $D'_\eta \subset D_\eta$ and $D'_0 \subset D_0$ be the unique chain of rational curves which forms Neron polygons. Then there are exactly n reduced components (resp. m reduced components) in D'_η (resp. D'_0), as it is of Kodaira type I_n (resp. I_m). Since $m > n$, there is a reduced irreducible component $E'_\eta \subset D'_\eta$ such that its specialization E'_0 has at least two reduced components. Let $\mathcal{E}' \subset \mathcal{D}$ be the closure of E'_η and $j : \tilde{\mathcal{E}} \rightarrow \mathcal{E}'$ the normalization. Let $\tilde{E}_0 := \tilde{\mathcal{E}} \times_S \{0\} = \sum_{j=1}^q r_j C_j$ be the special fiber.

$$\begin{array}{ccccc} \tilde{E}_\eta & \longrightarrow & \tilde{\mathcal{E}} & \longleftarrow & \tilde{E}_0 \\ \downarrow & & \downarrow j & & \downarrow \\ E'_\eta & \longrightarrow & \mathcal{E}' & \longleftarrow & E'_0 \\ \downarrow & & \downarrow & & \downarrow \\ X_\eta & \longrightarrow & \mathcal{X} & \longleftarrow & X_0 \end{array}$$

The generic fiber $\tilde{E}_\eta := \tilde{\mathcal{E}} \times_S \{\eta\}$ is a smooth irreducible rational curve. Let $T_1, T_2 \subset \tilde{\mathcal{E}}$ be the inverse image of the intersection locus of the Neron polygon D'_η . Let T_1 hits a component of \tilde{E}_0 , say C_1 . Then T_2 hits another component (say C_2), since the image $j(\tilde{E}_0) \subset D'_0$ has at least two component. One has $r_1 = r_2 = 1$, namely C_1 and C_2 are reduced components, since T_1 and T_2 are sections of $\tilde{\mathcal{E}} \rightarrow S$. Moreover the intersection points $(T_1 \cap C_1)$ and $(T_2 \cap C_2)$ are nonsingular points in $\tilde{\mathcal{E}}$.

Let f be a rational function on E_η such that the divisor $\text{div}_{E_\eta}(f) = T_1 - T_2$ and hence

$$\text{div}_{\tilde{\mathcal{E}}}(f) = T_1 - T_2 + \sum_{j=1}^q a_j C_j. \quad (2.3)$$

Then

$$\partial(Z_\eta) = \sum_{j=1}^q a_j C_j + N \cdot j(\tilde{E}_0) + (\text{other components of } D_0) \in \text{CH}^1(X_0)$$

for some $N \in \mathbb{Z}$ by the construction of Z_η . Therefore it is enough to show $a_1 \neq a_2$.

We take a desingularization $\rho : \mathcal{E} \rightarrow \tilde{\mathcal{E}}$ (if necessary). Then

$$\text{div}_{\mathcal{E}}(f) = T_1 - T_2 + \sum_{j=1}^q a_j C'_j + \sum_{j=q+1}^{\ell} a_j C'_j. \quad (2.4)$$

where C'_j ($j \leq q$) are the strict transform of C_j and C'_j ($j > q$) are the exceptional curves. We may assume T_i intersects with C'_i in \mathcal{E} for $i = 1, 2$, since $T_i \cap C_i$ are nonsingular points of $\tilde{\mathcal{E}}$. Renumbering C'_j , we may assume T_1 intersects with C'_1 and T_2 intersects with C'_ℓ . We then want to show $a_1 \neq a_\ell$. By the intersection theory, (2.4) yields

$$\begin{cases} 1 + \sum_{j=1}^{\ell} a_j (C'_1, C'_j) = 0 \\ \sum_{j=1}^{\ell} a_j (C'_i, C'_j) = 0 \end{cases} \quad 2 \leq \forall i \leq \ell - 1$$

namely

$$\begin{pmatrix} (C'_1, C'_1) & \cdots & (C'_1, C'_\ell) \\ \vdots & & \vdots \\ (C'_{\ell-1}, C'_1) & \cdots & (C'_{\ell-1}, C'_\ell) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_\ell \end{pmatrix} = \begin{pmatrix} -1 \\ \vdots \\ 0 \end{pmatrix} \quad (2.5)$$

By replacing (a_1, \dots, a_ℓ) with $(a_1, \dots, a_\ell) + c(r_1, \dots, r_\ell)$ for some $c \in \mathbb{Q}$, we may assume $a_\ell = 0$. Then

$$A \begin{pmatrix} a_1 \\ \vdots \\ a_{\ell-1} \end{pmatrix} = \begin{pmatrix} (C'_1, C'_1) & \cdots & (C'_1, C'_{\ell-1}) \\ \vdots & & \vdots \\ (C'_{\ell-1}, C'_1) & \cdots & (C'_{\ell-1}, C'_{\ell-1}) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{\ell-1} \end{pmatrix} = \begin{pmatrix} -1 \\ \vdots \\ 0 \end{pmatrix} \quad (2.6)$$

By Zariski's lemma ([2] III (8.2)), one has $\det A < 0$ and $\det A_{11} < 0$ where A_{11} is the cofactor matrix. Therefore one has $a_1 = -\det A_{11}/\det A \neq 0$, the desired assertion. This completes the proof. \square

2.3 Step 3 : Extension of admissible variations of MHS's

Let

$$\text{reg}(\mathcal{Z}^*) \in H_{\mathcal{D}}^3(\mathcal{X}^*, \mathbb{Q}(2))$$

be the Deligne-Beilinson cohomology class. Lemma 2.2 together with the commutative diagram

$$\begin{array}{ccccc} \text{CH}^2(\mathcal{X}^*, 1) & \xrightarrow{\partial} & \text{CH}^1(X_0) & & \\ \text{reg} \downarrow & & \downarrow cl & & \\ H_{\mathcal{D}}^3(\mathcal{X}^*, \mathbb{Q}(2)) & \xrightarrow{\text{Res}} & H_{\mathcal{D}}^2(X_0, \mathbb{Q}(1)) & \xrightarrow{\cong} & H^2(X_0, \mathbb{Q}(1)) \cap H^{1,1} \end{array} \quad (2.7)$$

yields nonvanishing $\text{reg}(\mathcal{Z}^*) \neq 0$. There is the isomorphism

$$H_{\mathcal{D}}^3(\mathcal{X}^*, \mathbb{Q}(2)) \cong \text{Ext}_{\text{VMHS}(S^*)}^1(\mathbb{Q}, H_{\mathbb{Q}}), \quad H_{\mathbb{Q}} := R^2\pi_*\mathbb{Q}(2) \quad (2.8)$$

with the extension group of admissible variations of mixed Hodge structures on S^* . Hence $\text{reg}(\mathcal{Z}^*)$ defines a non-trivial extension

$$0 \longrightarrow H_{\mathbb{Q}} \longrightarrow V_{\mathbb{Q}} \longrightarrow \mathbb{Q} \longrightarrow 0 \quad (2.9)$$

of admissible VMHS's on S^* . Tensoring with \mathbb{R} , one has an extension

$$0 \longrightarrow H_{\mathbb{R}} \longrightarrow V_{\mathbb{R}} \longrightarrow \mathbb{R} \longrightarrow 0 \quad (2.10)$$

of real VMHS's. This is also a non-trivial extension. Then the following lemma finishes the proof of Theorem 3.1.

Lemma 2.3 *Let $\Delta \subset S$ be a small neighborhood of 0, and put $\Delta^* = \Delta \setminus \{0\}$. For $t \in \Delta^*$ such that $0 < |t| \ll 1$, the extension*

$$0 \longrightarrow H_{\mathbb{R},t} \longrightarrow V_{\mathbb{R},t} \longrightarrow \mathbb{R} \longrightarrow 0 \quad (2.11)$$

is non-trivial where $H_{\mathbb{R},t}$ etc. denotes the fiber at t . Hence the nonvanishing (2.2) follows.

Before proving Lemma 2.3, we note that “ \mathbb{Q} -regulator indecomposability of Z_t ” is immediate from the fact that (2.9) is non-trivial. Let $H_{\mathcal{O}} := \mathcal{O}_{S^*} \otimes H_{\mathbb{Q}}$ and F^{\bullet} be the Hodge bundles. Put $J_{\mathbb{Q}} := H_{\mathcal{O}}/(F^2 + H_{\mathbb{Q}})$ and

$$J_{\mathbb{Q}}^h := \text{Ker}[J_{\mathbb{Q}} \xrightarrow{\nabla} \Omega_{S^*}^1 \otimes H_{\mathcal{O}}/F^1]$$

the sheaf of horizontal sections where ∇ denotes the Gauss-Manin connection. As is well-known, there is the injective map

$$\iota : \text{Ext}_{\text{VMHS}(S^*)}^1(\mathbb{Q}, H_{\mathbb{Q}}) \hookrightarrow \Gamma(S^*, J_{\mathbb{Q}}^h), \quad (2.12)$$

and $\nu_{\mathcal{Z}^*} := \iota(\text{reg}(\mathcal{Z}^*)) \in \Gamma(S^*, J_{\mathbb{Q}}^h)$ is called the *normal function* associated to \mathcal{Z}^* . Since the zero locus of the normal function is at most a countable set, we have $\nu_{\mathcal{Z}^*}(t) \neq 0$ for a general t , and hence that Z_t is \mathbb{Q} -regulator indecomposable. However to obtain the “ \mathbb{R} -regulator indecomposability” in the same way, we need to show that the injectivity of (2.12) remains true if we replace $J_{\mathbb{Q}}^h$ with $J_{\mathbb{R}}^h$, and I don't know how to prove it in general. We prove Lemma 2.3 in a different way.

Proof of Lemma 2.3.

Let T be the local monodromy around Δ^* . The action on $H_t = H^2(X_t, \mathbb{Q}(2))$ is trivial, whereas that on $V_{\mathbb{Q},t}$ is non-trivial. Indeed, let $\{e_{1,t}, \dots, e_{m,t}\}$ be a basis of $H_{\mathbb{Q},t}$ and $\{e_{0,t}, e_{i,t}\}$ be a basis of $V_{\mathbb{Q},t}$ such that $e_{0,t} \notin H_{\mathbb{Q},t}$. Let $N = T - 1$ be the log monodromy. Then one has $(2\pi i)^{-1} N e_{0,t} = \text{Res}(\text{reg}(\mathcal{Z}^*)) = \text{cl}[\partial \mathcal{Z}^*] \neq 0$ under the natural isomorphism $H_{\mathbb{Q},t} \otimes \mathbb{Q}(-1) \cong H_{\mathbb{Q},0} \otimes \mathbb{Q}(-1) = H^2(X_0, \mathbb{Q}(1))$.

Let us fix a frame $z_t \in H_{\mathbb{Q},t}$ which satisfies that $z|_{t=0} = cl[\partial \mathcal{Z}^*] \in H^2(X_0, \mathbb{C})$. Then the *admissibility* of V yields that there are holomorphic functions $u_i(t)$ on Δ^* with at most meromorphic singularities at $t = 0$ such that

$$f_0 = e_{0,t} + \frac{\log(t)}{2\pi i} z_t + \sum_i u_i(t) e_{i,t}$$

belongs to the Hodge bundle $\Gamma(\Delta^*, F^0 V_{\mathcal{O}})$. Moreover, by adding some $\theta \in \Gamma(\Delta^*, F^2 H_{\mathcal{O}})$, one has

$$f'_0 = f_0 + \theta = e_{0,t} + \frac{\log(t)}{2\pi i} z_t + \sum_i u'_i(t) e_{i,t}$$

with $u'_i(t)$ holomorphic at $t = 0$. Therefore the extension data of (2.9) is given as follows

$$(\text{extension class of } V|_{\Delta^*}) = \frac{\log(t)}{2\pi i} z_t + \sum_i u'_i(t) e_{i,t} \in H_{\mathcal{O}}/(F^2 + H_{\mathbb{Q}}). \quad (2.13)$$

Let \mathcal{A}_{Δ^*} be the sheaf of C^∞ -functions on Δ^* , and let $c : H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{A}_{\Delta^*} \rightarrow H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{A}_{\Delta^*}$ the complex conjugation given by $c(x \otimes f) = x \otimes \bar{f}$. Let

$$H_{\mathcal{O}}/(F^2 + H_{\mathbb{R}}) \xrightarrow{1-c} H_{\mathbb{R}} \otimes \mathcal{A}_{\Delta^*}/(1-c)F^2 \cong (H_{\mathbb{R}} \cap H^{1,1}) \otimes \mathcal{A}_{\Delta^*}. \quad (2.14)$$

Then the extension class (2.13) goes to

$$\frac{\log |t|}{\pi} z_t + \sum_i (\text{bounded function}) e_{i,t} \in (H_{\mathbb{R}} \cap H^{1,1}) \otimes \mathcal{A}_{\Delta^*} \quad (2.15)$$

via (2.14). This does not vanish for $0 < |t| \ll 1$, and hence the proof is done. \square

This completes the proof of Theorem 1.1.

3 Example : Self-Product of elliptic curves

Recall the following theorem due to Lewis and Gordon ¹.

Theorem 3.1 (Lewis - Gordon, [4] Thm.1) *Let $X = E \times E$ be a product of elliptic curves over \mathbb{C} . If E is sufficiently general, then there exists a \mathbb{R} -regulator indecomposable cycle $z \in \text{CH}^2(X, 1)$ ².*

We here give an alternative proof as an application of Theorem 1.1. Let

$$E_a : y^2 = F(x) = x(x-1)(x-a), \quad a \in \mathbb{C} \setminus \{0, 1\}$$

¹Though there was an error in their proof, Türkmen recently corrected it [7].

² [4] Thm.1 deals with a general product $E_1 \times E_2$, though their techniques allow to handle the case of a self-product $E \times E$ as well.

be an elliptic curve over \mathbb{C} . Let $X_a := \text{Km}(E_a \times E_a)$ be the *Kummer surface* associated to the product $E_a \times E_a$. The birational model of X_a is given by the double sextic

$$X_a : w^2 = F(x_1)F(x_2) = x_1x_2(x_1 - 1)(x_2 - 1)(x_1 - a)(x_2 - a).$$

Changing the variables $x := x_1$, $t := x_1x_2$ and $y := wx_1$, we get

$$X_a : y^2 = t(x - 1)(x - t)(x - a)(ax - t). \quad (3.1)$$

This gives an elliptic fibration

$$f : X_a \longrightarrow \mathbb{P}^1, \quad (x, y, t) \longmapsto t \quad (3.2)$$

where t is the affine parameter of \mathbb{P}^1 . This is naturally extended to a family over $S = \mathbb{A}_{\mathbb{C}} \setminus \{0, 1\}$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathbb{P}^1 \times S \\ & \searrow \pi & \swarrow \\ & S & \end{array}$$

in which $X_a = \pi^{-1}(a)$. We fix a section s of $(x, y) = (1, 0)$.

Let us look at the singular fibers in (3.2). The singular fibers are located at $t = 1, a, a^2, 0, \infty$. Two additive fibers appear at $t = 0, \infty$ and both are of Kodaira type I_2^* . If $a \neq -1$, the singular fibers at $t = 1, a, a^2$ are multiplicative of type I_2, I_4, I_2 respectively, and if $a = -1$, two fibers at $t = \pm 1$ are multiplicative of type I_4 . Let $\text{NF}(X_a) \subset \text{NS}(X_a)$ be the subgroup generated by irreducible components of singular fibers and the section s . The rank of $\text{NF}(X_a)$ is 19 if $a \neq -1$, and 20 if $a = -1$. As is well-known,

$$\text{rankNS}(X_a) = \begin{cases} 19 & E_a \text{ has no CM} \\ 20 & E_a \text{ has a CM.} \end{cases} \quad (3.3)$$

In particular

$$\text{NF}(X_a) \otimes \mathbb{Q} = \text{NS}(X_a) \otimes \mathbb{Q} \text{ if and only if } E_a \text{ has no CM.} \quad (3.4)$$

If $a \neq -1$, there is the isomorphism

$$H^2(X_a, \mathbb{Q})/\text{NF}(X_a)_{\mathbb{Q}} \cong \text{Sym}^2 H^1(E_a, \mathbb{Q}). \quad (3.5)$$

We now apply Theorem 1.1 for $\mathcal{D} = f^{-1}(1)$ and $X_{-1} = \pi^{-1}(-1)$. As we see in the above, D_{η} is a split multiplicative fiber of type I_2 and $D_{-1} := \mathcal{D} \times_{\mathcal{X}} X_{-1}$ is of type I_4 , so the conditions (1) and (2) are satisfied. We obtain a higher Chow cycle $Z_a \in \text{CH}^2(X_a, 1)$ arising from $f^{-1}(1)$, and this is \mathbb{R} -regulator indecomposable for a general a by Thm. 1.1 and (3.4). By (3.5), this gives a \mathbb{R} -regulator indecomposable cycle in a self product $E_a \times E_a$.

Remark 3.2 The real regulator $\text{reg}(Z_a)$ (as a function of a) is studied in detail in [3] §6.4.

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